

Histories quantisation of parameterised systems: I. Development of a general algorithm

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Abstract

We develop a new algorithm for the quantisation of systems with first-class constraints. Our approach lies within the (History Projection Operator) continuous-time histories quantisation programme. In particular, the Hamiltonian treatment (either classical or quantum) of parameterised systems is characterised by the loss of the notion of time in the space of true degrees of freedom (*i.e.* the ‘problem of time’). The novel temporal structure of the HPO theory (two laws of time transformation that distinguish between the temporal logical structure and the dynamics) persists after the imposition of the constraints, hence the problem of time does not arise. We expound the algorithm for both the classical and quantum cases and apply it to simple models.

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I Introduction

I.1 Preamble

In this paper we propose an algorithm for both the classical and quantum treatments of constrained systems, based on the notion of consistent histories. The scheme lies within the general framework of the consistent histories approach to quantum theory [1, 2, 3, 4]. More precisely, we generalise and extend the results of the HPO (History Projection Operator) continuous-time histories programme [5, 6, 7, 8, 9, 10] to incorporate the quantisation of systems with first-class constraints. In particular, we shall see how these methods bring substantially new insights for dealing with parameterised systems.

The Hamiltonian treatment of (first-class) constrained systems is well understood. In the classical case the aim is to construct the reduced phase space, *i.e.*, a symplectic manifold on which the true degrees of freedom are defined, and then consider the time evolution induced by the Hamiltonian. The quantum treatment can proceed with reduced phase space quantisation, where one employs the standard quantisation algorithms for the classical theory on this space. Alternatively, one can use a Dirac type of quantisation where the Hilbert space of the unconstrained system is constructed and then the constraint is implemented quantum mechanically by finding the projector onto the *physical subspace* of the Hilbert space.

However for a particular class of system, namely the parameterised systems, the canonical treatment leads to some deep conceptual problems. Parameterised systems have a vanishing Hamiltonian when the constraints are imposed. Classically this implies that the elements of the reduced phase space are themselves *solutions to the classical equations of motion*, hence they define a classical history of the system. On the other hand, a point of phase space corresponds to a possible configuration of the physical system at an instant of time. As a result, the notion of time is unclear, or even ambiguous, in these systems: in particular, it is not obvious how to recover the notion of temporal ordering unless we choose to arbitrarily impose a gauge-fixing condition. This is also true at the quantum level. There are no dynamics in the physical Hilbert space because the Hamiltonian operator vanishes; the system seems to be in a ‘frozen’ state. This is one facet of the “problem of time” for parameterised systems, and is particularly acute in the Hamiltonian (or path integral) treatment of general relativity (see [11, 12] for a thorough review).

The HPO continuous-time histories algorithm is particularly suited to deal with systems that have a non-trivial temporal structure. This is due to the striking property that HPO histories admit two distinct laws of time transformation, each representing a distinct quality of time [9]. The first corresponds to time considered purely as a kinematical parameter of a physical system, with respect to which a history is defined as a succession of possible events. It is strongly connected with the temporal-logical structure of the theory and is related to the view of time as a parameter that determines the ordering of events. The second mode in which time is manifested in this approach, corresponds to the

dynamical evolution generated by the Hamiltonian. Classically these two laws are nicely intertwined through the action principle which provides the paths that are solutions to the classical equations of motion. For a detailed presentation of the HPO continuous-time programme see [13].

The fact that histories describe objects that have an intrinsic temporality, together with the two notions of time transformation, enables us to treat parameterised systems in such a way that the problem of time does not arise. As we shall see, HPO histories keep their intrinsic temporality after the implementation of the constraint, thus there is no uncertainty about the temporal-ordering properties of the physical system.

Another important result is that the theory admits time reparameterisation as a symmetry: classically it arises as an invariance of the equations of motion, while in quantum theory it is an invariance of the assignments of probabilities.

This has been a very brief description of the aims and results of this paper. We shall now give some more details about the underlying concepts and ideas of the histories quantisation scheme.

I.2 Consistent histories

In classical Newtonian theory, time is introduced as an external parameter; and in all the existing approaches to quantum theory, the treatment of time is inherited from the classical theory. On the other hand, general relativity treats time as an internal parameter of the theory: in particular, it is one of the coordinates of the spacetime manifold. When we attempt to combine the two theories in quantum gravity, this essential difference in the treatment of time appears as a major problem—one of the aspects of what is known as the ‘Problem of Time’. One of the directions towards a solution of the problem is to construct theories where time is introduced in a novel way.

One such formalism is the consistent histories approach to quantum theory in which time appears as the label on a time-ordered sequence of projection operators which represents a ‘history’ of the system. In the original scheme by Gell-Mann and Hartle [1, 2, 3], the crucial object is the decoherence functional written as

$$d(\alpha, \beta) = \text{tr}(\tilde{C}_\alpha^\dagger \rho \tilde{C}_\beta) \quad (\text{I. 1})$$

where ρ is the initial density-matrix, and where the *class operator* \tilde{C}_α is defined in terms of the standard Schrödinger-picture projection operators α_{t_i} as

$$\tilde{C}_\alpha := U(t_0, t_1)\alpha_{t_1}U(t_1, t_2)\alpha_{t_2}\dots U(t_{n-1}, t_n)\alpha_{t_n}U(t_n, t_0) \quad (\text{I. 2})$$

where $U(t, t') = e^{-i(t-t')H/}$ is the unitary time-evolution operator from time t to t' . Each projection operator α_{t_i} represents a proposition about the system at time t_i , and the class operator \tilde{C}_α represents the composite history proposition “ α_{t_1} is true at time t_1 , and then α_{t_2} is true at time t_2 , and then \dots , and then α_{t_n} is true at time t_n ”.

The consistent histories approach allows the description of an approximately classical domain emerging from the macroscopic behaviour of a closed physical

system, as well as its microscopic properties, in terms of the conventional Copenhagen quantum mechanics. This is possible through the decoherence condition: the requirement for ‘decoherence’ (negligible interference between disjoint histories) selects a consistent set of histories that can be represented on a classical (Boolean) lattice, thus having a classical logical structure. Hence in the consistent histories theory, emphasis is given to the observation that, although in atomic scales a system is described by quantum mechanics, it may also be described by classical mechanics and ordinary logic. Therefore a more refined logical structure seems to be a necessary part of any consistent histories formalism. However, the Gell-Mann and Hartle approach lacks the logical structure of standard quantum mechanics in the sense that the fundamental entity (*i.e.*, history) for the description of the system is not represented by a projector in the Hilbert space of the standard theory. This is because, as a product of (generically, non-commuting) projection operators, the class-operator \hat{C}_α , that represents a history, is not itself a projector.

I.3 The History Projection Operator Approach

The difference between the representation of propositions in standard quantum mechanics and in the history theory is resolved in the alternative approach of the ‘History Projection Operator’ theory [5, 6], in which the history proposition “ α_{t_1} is true at time t_1 , and then α_{t_2} is true at time t_2 , and then \dots , and then α_{t_n} is true at time t_n ” is represented by the *tensor product* $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \dots \otimes \alpha_{t_n}$ which, unlike \hat{C}_α , is a genuine projection operator, albeit one that is defined on the tensor product of copies of the standard Hilbert space $\mathcal{V}_n = \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \dots \otimes \mathcal{H}_{t_n}$. Hence the ‘History Projection Operator’ formalism extends to multiple times, the quantum logic of single-time quantum theory.

However, the introduction of the tensor product $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \dots \otimes \mathcal{H}_{t_n}$ led to a quantum theory where the notion of time appears mainly via its partial ordering property (quasi-temporal behaviour). In particular, there was no natural way to express the time translations from one time slot—that refers to one copy of the Hilbert space \mathcal{H}_t —to another one, that refers to another copy $\mathcal{H}_{t'}$. The situation changed when the *the continuous limit* of such tensor products was introduced: hence forward, time appears uniformly in a continuous way.

One of the original problems in the development of the HPO theory was the lack of a clear physical meaning of some of the quantities involved. The introduction of the history group by Isham and Linden [7] made a significant step in this direction in the sense that the spectral projectors of the history Lie algebra represent propositions about the appropriate phase space observables of the system.

In standard canonical quantum theory it has been argued that the identification of Hilbert space and observables can come from the study of the representations of the canonical group: a group of canonical transformations that acts transitively on the phase space of the classical system [15]. For the case of a particle moving on the real line \mathbb{R} , the Hilbert space \mathcal{H} of the canonical

theory carries a representation of the Heisenberg-Weyl group with Lie algebra

$$[x, p] = i\hbar. \quad (\text{I. 3})$$

In the history theory, the tensor product \mathcal{V}_n of copies of the standard Hilbert space carries a unitary representation of the n -fold product group whose generators satisfy

$$[x_{t_i}, x_{t_j}] = 0 \quad (\text{I. 4})$$

$$[p_{t_i}, p_{t_j}] = 0 \quad (\text{I. 5})$$

$$[x_{t_i}, p_{t_j}] = i\hbar\delta_{ij} \quad (\text{I. 6})$$

where $\{t_1, t_2, \dots, t_n\}$ is a discrete subset of the real line containing the instants of time at which the propositions are asserted. Thus, the tensor product \mathcal{V}_n can be viewed as arising as a representation space for Eqs. (I. 4)–(I. 6), and the tensor products $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \dots \otimes \alpha_{t_n}$ —that correspond to sequential histories about the values of position or momentum (or linear combinations of them)—are then elements of the spectral representations of this Lie algebra.

In the case of a continuous time label, it is natural to postulate a continuous analogue of the history algebra Eqs. (I. 4)–(I. 6) in which the Kronecker delta on the right hand side of Eq. (I. 6) is replaced by a Dirac delta function. This has the striking consequence that the history propositions about the system are intrinsically time-averaged quantities: this means that the physical quantities cannot be defined at *sharp moments in time*. The continuous-time histories formalism is the basis of the work that follows.

For purposes of clarity, we have found it useful to briefly review the continuous-time histories programme. The main ideas are elaborated in section II together with an explanation of the details of the general construction. Particular emphasis is placed on the two distinct laws of time transformation that arise in history theories, and their physical interpretation.

In section III we proceed to examine the classical histories treatment of parameterised systems. We show that given a constraint $h = 0$ in the canonical theory we can write the corresponding function H_κ in the history theory. The symplectic transformations generated by H_κ partition the space of histories into orbits and allow us to define the space of reduced phase space histories. This has the *same temporal structure* as the initial space of histories and hence a clear ordering structure for time, even for parameterised systems. We will show that the solutions to the classical equations of motion are reparameterisation invariant. As an example we study the system of two harmonic oscillators with a constant energy difference.

In section IV we proceed to give the quantisation algorithm. A Hilbert space is constructed by seeking a representation of the history group in which the Liouville and Hamiltonian functions are represented by self-adjoint operators. The constraint is implemented by projecting any operator onto the subspace where the quantum version of the constraints are satisfied. The decoherence functional is explicitly constructed and its invariance under time reparameterisation estab-

lished. The example of section III is now quantised and the algorithm is readily implemented since the constraint operator has a discrete spectrum.

Section V gives a preliminary account of the case of a relativistic particle. The canonical constraint has a continuous spectrum, and hence the quantisation algorithm has to be modified for this case. We discuss briefly the strategy to be followed in a subsequent paper in which we shall deal with this larger class of systems.

Finally in section VI we discuss and review our results, putting emphasis on their relevance for the study of quantum gravity.

We should remark that while in this paper we focus primarily on parameterised systems, the algorithm we present is valid for any system with first class constraints.

II The continuous-time histories programme

As was mentioned in the Introduction, the history group was first introduced for discrete-time histories [7], and was then developed to the continuous time case by introducing a delta function in the unequal-time history commutation relations. As an immediate consequence, an intriguing feature of the theory appeared: that all interesting history propositions are about *time-averaged* physical quantities.

II.1 The History Space

The History Group. To discuss continuous-time histories we employ the same approach as the one described in the Introduction. Thus, motivated by Eqs. (I. 4)–(I. 6), we started with the history-group whose Lie algebra (History Algebra) is [7]

$$[x_{t_1}, x_{t_2}] = 0 \quad (\text{II. 1})$$

$$[p_{t_1}, p_{t_2}] = 0 \quad (\text{II. 2})$$

$$[x_{t_1}, p_{t_2}] = i\hbar\tau\delta(t_1 - t_2) \quad (\text{II. 3})$$

where $-\infty \leq t_1, t_2 \leq \infty$; the constant τ has dimensions of *time* [9]. It does not appear in any physical quantities; henceforward we shall set $\hbar\tau = 1$. Note that these operators are in the *Schrödinger* picture.

The choice of the Dirac delta function in the right hand side of Eq. (II. 3) is closely associated with the requirement that time be treated as a continuous variable. As emphasised earlier, one consequence is the fact that the observables cannot be defined at sharp moments of time but rather as time-averaged quantities.

We note that Eqs. (II. 1)–(II. 3) are mathematically the same as the canonical commutation relations of a quantum *field* theory in one space dimension. In particular, in order that equations of the type Eqs. (II. 1)–(II. 3) are mathematically well-defined they must be smeared with test functions to give

$$[x_f, x_g] = 0 \quad (\text{II. 4})$$

$$[p_f, p_g] = 0 \quad (\text{II. 5})$$

$$[x_f, p_g] = i \int_{-\infty}^{\infty} f(t)g(t) dt, \quad (\text{II. 6})$$

where the class of test functions s is a linear subspace of the space $L^2(\mathbb{R}, dt)$ of square integrable functions on \mathbb{R} . The physically appropriate representation of the HA Eqs. (II. 4)–(II. 6), —bearing in mind that infinitely many unitarily inequivalent representations are known to exist in the analogous case of quantum field theory—can be uniquely selected by the requirement that the time-averaged energy exists as a proper self-adjoint operator [14, 8].

Fock representation. The selection of the representation by demanding the existence of a time-averaged energy can be explicitly carried out for quadratic systems. Here we shall examine the example of the one-dimensional, simple harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2. \quad (\text{II. 7})$$

We expect to have a one-parameter family of operators that represent the energy at time t

$$H_t := \frac{p_t^2}{2m} + \frac{m\omega^2}{2}x_t^2 \quad (\text{II. 8})$$

More precisely, we are interested in a time-averaged Hamiltonian operator H_κ , which is defined heuristically as $H_\kappa = \int_{-\infty}^{\infty} dt \kappa(t) H_t$.

We shall select a Fock representation for the history algebra that has the structure of a continuous tensor product. The Fock space is defined as $\mathcal{V} = e^{L^2(\mathbb{R}, dt)}$, and the generators of the history group are

$$x_t = \left(\frac{1}{2\omega}\right)^{1/2}(b_t + b_t^\dagger) \quad (\text{II. 9})$$

$$p_t = i\left(\frac{\omega}{2}\right)^{1/2}(b_t^\dagger - b_t) \quad (\text{II. 10})$$

in terms of the annihilation and creation operators b_t and b_t^\dagger on the Fock space. Then the operator H_κ can be shown to exist for all measurable functions $\kappa(t)$, since the automorphisms

$$e^{iH_\kappa s} b_t e^{-iH_\kappa s} = e^{-i\omega\kappa(t)s} b_t \quad (\text{II. 11})$$

are unitarily implementable.

The Fock space contains the unnormalised coherent states $|\exp w(\cdot)\rangle_{\mathcal{V}}$ where $w(\cdot) \in L^2(\mathbb{R}, dt)$. If one compares these to the unnormalised coherent states $|\exp w\rangle$ of the single-time Hilbert space $H_t = e^{\mathbb{C}}$ on which the canonical group is represented, we get the fundamental automorphism

$$\begin{aligned} \otimes_t H_t &= \mathcal{V} \\ \otimes_t |\exp w_t\rangle_{H_t} &= |\exp w(\cdot)\rangle_{\mathcal{V}}. \end{aligned} \quad (\text{II. 12})$$

The Action and the Liouville operators. The question how the Schrödinger-picture objects with different time labels—like the history algebra generators x_t and p_t —are related was addressed in [9] by studying the time transformation laws of the theory. The fact that the Hamilton-Jacobi functional S , evaluated on the realised path of the system—*i.e.*, for a solution of the classical equations of motion, under some initial conditions—is the generating function of a canonical transformation which transforms the system variables, led to the definition of the action operator

$$S_\kappa := \int_{-\infty}^{+\infty} (p_t \dot{x}_t - \kappa(t) H_t) dt \quad (\text{II. 13})$$

which can be defined rigorously through its automorphisms [9]

$$e^{isS_\kappa} b_t e^{-isS_\kappa} = e^{i\omega \int_t^{t+s} \kappa(t+s') ds'} e^{s \frac{d}{dt}} b_t. \quad (\text{II. 14})$$

The first term of the action operator eq. (II. 13) is identical to the kinematical part of the classical phase space action functional. This ‘Liouville’ operator is formally written as

$$V := \int_{-\infty}^{\infty} (p_t \dot{x}_t) dt \quad (\text{II. 15})$$

so that

$$S_\kappa = V - H_\kappa \quad (\text{II. 16})$$

The Liouville operator is rigorously defined through its automorphisms

$$e^{isV} b_t e^{-isV} = b_{t+s}. \quad (\text{II. 17})$$

General systems. This construction can be generalised for more general systems with quadratic Hamiltonians. If the canonical commutation relations are smeared by elements of a real vector space V , then the corresponding Hilbert space of the canonical theory is the Fock space $H = e^{V_c}$, where $V_c = V \otimes \mathbb{C}$. Unnormalised coherent states $|\exp w\rangle$ with $w \in V_c$ are then naturally defined on H .

The corresponding Weyl group for histories has as its space of smearing functions the Hilbert space $L^2(\mathbb{R}, dt)_{\mathbb{R}} \otimes V$ and can be represented in the history Fock space $\mathcal{V} = e^{L^2(\mathbb{R}, dt) \otimes V_c}$. This Fock space contains the natural unnormalised coherent states $|\exp w(\cdot)\rangle$ with $w(\cdot) \in L^2(\mathbb{R}, dt) \otimes V_c$ and there is an isomorphism

$$\begin{aligned} \otimes_t (e^{V_c})_t &= e^{L^2(\mathbb{R}, dt) \otimes V_c} \\ \otimes_t |\exp w_t\rangle_{H_t} &= |\exp w(\cdot)\rangle_{\mathcal{V}} \end{aligned} \quad (\text{II. 18})$$

For more general Hamiltonians, we can no longer rely on the Fock type of representations. The selection of a representation is particularly difficult when the Hamiltonian has only continuous spectrum. This will be a problem in the quantisation of the relativistic particle.

II.2 The temporal structure

We have seen that the Liouville and the Hamilton operators generate two distinct laws of time transformation. This is a fundamental property of history theories. In this section we shall explain their physical meaning [9, 13].

In the continuous-time histories theory, the Hamiltonian operator H_t produces phase changes in time, preserving the time label t of the Hilbert space on which, at least formally, H_t is defined. On the other hand, and analogous to the classical case, it is the Liouville operator V that assigns history commutation relations, and produces time transformations ‘from one Hilbert space to another’. The action operator generates a combination of these two types of time transformation.

This is easier to see through the definition of a Heisenberg-picture analogue of x_t

$$x_{\kappa,t,s} : = e^{isH_\kappa} x_t e^{-isH_\kappa} \quad (\text{II. 19})$$

$$= \cos[\omega s \kappa(t)] x_t + \frac{1}{m\omega} \sin[\omega s \kappa(t)] p_t \quad (\text{II. 20})$$

and the similar definition for $p_{\kappa,t,s}$.

If we use the notation $x_f(s)$ for the *history* Heisenberg-picture operators smeared with respect to the time label t Eq. (II. 20) (with $\kappa = 1$), we observe that they behave as standard Heisenberg-picture operators, with time parameter s .

We now define a one-parameter group of transformations $T_V(\tau)$, with elements $e^{i\tau V}$, $\tau \in \mathbf{R}$ where V is the Liouville operator Eq. (II. 15), and we consider its action on the Heisenberg-picture operator $b_{t,s}$ (for simplicity we write the unsmeared expressions)

$$e^{i\tau V} b_{t,s} e^{-i\tau V} = b_{t+\tau,s}, \quad (\text{II. 21})$$

which makes particularly clear the sense in which the Liouville operator is the generator of transformations of the time parameter t labelling the Hilbert spaces \mathcal{H}_t .

Next we define a one-parameter group of transformations $T_H(\tau)$, with elements $e^{i\tau H}$, where H is the Hamiltonian operator smeared with the function $\kappa(t) = 1$

$$e^{i\tau H} b_{t,s} e^{-i\tau H} = b_{t,s+\tau}. \quad (\text{II. 22})$$

Thus the Hamiltonian operator is the generator of phase changes of the time parameter s , produced only on one Hilbert space \mathcal{H}_t , for a fixed value of the ‘external’ time parameter t .

Finally, we define the one-parameter group of transformations $T_S(\tau)$, with elements $e^{i\tau S}$, where S is the action operator, which acts as

$$e^{i\tau S} b_{t,s} e^{-i\tau S} = b_{t+\tau,s+\tau} \quad (\text{II. 23})$$

We see that the action operator generates both types of time transformations—a feature that appears only in the HPO scheme.

In standard quantum theory, time evolution is described by two different laws: the state-vector reduction that supposedly occurs when a measurement is made, and the unitary time-evolution that takes place between measurements.

It is our contention that the two types of time-transformation observed in the continuous-time histories are associated with the two dynamical processes in standard quantum theory: the time transformations generated by the Liouville operator V are related to the causal ordering implied by the temporal logic nature of the histories construction (we shall argue in section II.4 that it is related to the state-vector reduction in the canonical theory), while the time transformations produced by the Hamiltonian operator H are related to the unitary time-evolution between “measurements”.

The Hamiltonian operator, which produces transformations via a type of Heisenberg time-evolution, appears as the ‘clock’ of the theory. As such, it depends on the particular physical system that the Hamiltonian describes. Indeed, we would expect the definition of a ‘clock’ for the evolution in time of a physical system to be connected with the dynamics of the system concerned.

We note that the smearing function $\kappa(t)$ used in the definition of the Hamiltonian operator can be interpreted as a mechanism for implementing the idea of reparameterising time; in the present context however, κ is kept fixed for a particular physical system.

The coexistence of the two types of time-evolution, as reflected in the action operator identified as the generator of such time transformations, is a striking result. In particular, its definition is in accord with the classical analogue, namely the Hamilton action functional. In classical theory, a distinction also arise between a kinematical and a dynamical part of the action functional in the sense that the first part corresponds to the symplectic structure and the second to the Hamiltonian.

II.3 The classical histories

Let us consider the space of classical histories Π viewed as the set of smooth¹ paths on the classical phase space Γ . Hence an element of Π is a path $\gamma : \mathbb{R} \rightarrow \Gamma$. For our purposes we shall have to define vector fields and differential forms on Π . Therefore one has to equip Π with the structure of tangent and cotangent spaces at one point. To avoid the need to introduce a specific (infinite-dimensional) differential structure on Π , we use the standard trick of employing the following definitions. The tangent space $T_\gamma \Pi$ at $\gamma \in \Pi$ is defined as

$$T_\gamma \Pi = \{v : \mathbb{R} \rightarrow T\Gamma \mid v(t) \in T_{\gamma(t)}\Gamma\} \quad (\text{II. 24})$$

where $T\Gamma$ is the tangent bundle of Γ . Correspondingly the cotangent space at γ is defined as

$$T_\gamma^* \Pi = \{\omega : \mathbb{R} \rightarrow T^*\Gamma \mid \omega(t) \in T_{\gamma(t)}^*\Gamma\} \quad (\text{II. 25})$$

¹The requirement of smoothness is made for simplicity, but it might be appropriate to extend the paths to a larger class of function for certain purposes; for example, in a careful discussion of the temporal logic encoded in the classical theory.

and the pairing between covariant and contravariant vectors at the point $\gamma \in \Pi$ is given by

$$\langle \omega(\cdot), v(\cdot) \rangle_\gamma = \int dt \langle \omega(t), v(t) \rangle_{\gamma(t)} \quad (\text{II. 26})$$

where the brackets within the integral denote the standard pairing on Γ (thus the functions $v(\cdot)$ and $\omega(\cdot)$ must be such that this integral exists).

Taking for simplicity $\Gamma = \mathbb{R} \times \mathbb{R} = \{(x, p)\}$, we can define x_t and p_t as functions on Π by

$$x_t(\gamma) := x(\gamma(t)) \quad (\text{II. 27})$$

$$p_t(\gamma) := p(\gamma(t)) \quad (\text{II. 28})$$

In general given a function f on Γ we can define a family $t \mapsto F_t$ of functions on Π as

$$F_t(\gamma) := f(\gamma(t)) \quad (\text{II. 29})$$

We can equip Π with a symplectic form

$$\omega = \int dt dp_t \wedge dx_t \quad (\text{II. 30})$$

which generates a Poisson bracket

$$\{x_t, p_{t'}\}_\Pi = \delta(t, t'). \quad (\text{II. 31})$$

More generally, for two families of functions $t \mapsto F_t$ and $t \mapsto G_t$ defined through (II. 29) we have

$$\{F_t, G_{t'}\} = L_t \delta(t, t'). \quad (\text{II. 32})$$

where L_t corresponds to the function l on Γ

$$l = \{f, g\}_\Gamma. \quad (\text{II. 33})$$

We now define the phase space action functional $S(\gamma)$ on Π as

$$S(\gamma) := \int [p_t \dot{x}_t - H_t(p_t, x_t)](\gamma) dt \quad (\text{II. 34})$$

where $\dot{x}_t(\gamma)$ is the velocity at the time point t of the path γ .

We also define the history classical analogues for the Liouville and time-averaged Hamiltonian operators as

$$V(\gamma) := \int [p_t \dot{x}_t](\gamma) dt \quad (\text{II. 35})$$

$$H(\gamma) := \int [H_t(p_t, x_t)](\gamma) dt \quad (\text{II. 36})$$

$$S(\gamma) = V(\gamma) - H(\gamma) \quad (\text{II. 37})$$

In classical mechanics, the least action principle states that there exists a functional $S(\gamma) = \int [p\dot{x} - H(p, x)](\gamma) dt$ such that the physically realised path

is a curve in state space, γ_0 , with respect to which the condition $\delta S(\gamma_0) = 0$ holds, when we consider variations around this curve. From this, the Hamilton equations of motion are deduced to be

$$\dot{x} = \{x, H\} \quad (\text{II. 38})$$

$$\dot{p} = \{p, H\} \quad (\text{II. 39})$$

where x and p —the coordinates of the realised path γ_0 —are the solutions of the classical equations of motion [9, 13]. For any function $F(x, p)$ of the classical solutions it is also true that

$$\{F, H\} = \dot{F} \quad (\text{II. 40})$$

In the case of classical continuous-time histories, one can formulate the above variational principal in terms of the Hamilton equations with the statement: A classical history γ_{cl} is the realised path of the system—*i.e.* a solution of the equations of motion of the system—if it satisfies the equations

$$\{x_t, V\}(\gamma_{cl}) = \{x_t, H\}(\gamma_{cl}) \quad (\text{II. 41})$$

$$\{p_t, V\}(\gamma_{cl}) = \{p_t, H\}(\gamma_{cl}) \quad (\text{II. 42})$$

where $\gamma_{cl} = t \mapsto (x_t(\gamma_{cl}), p_t(\gamma_{cl}))$, and $x_t(\gamma_{cl})$ is the position coordinate of the realised path γ_{cl} at the time point t . The eqs. (II. 41–II. 42) are the history equivalent of the Hamilton equations of motion.

One would have expected this result for the classical analogue of the histories formalism, as it shows that the classical analogue of the two types of time-transformation in the quantum theory coincide.

From the eqs. (II. 41–II. 42) we also conclude that the canonical transformation generated by the history action functional $S(\gamma)$, leaves invariant the paths that are classical solutions of the system:

$$\{x_t, S\}(\gamma_{cl}) = 0 \quad (\text{II. 43})$$

$$\{p_t, S\}(\gamma_{cl}) = 0 \quad (\text{II. 44})$$

It is also the case that any function F on Π satisfies the equation

$$\{F, S\}(\gamma_{cl}) = 0. \quad (\text{II. 45})$$

II.4 The general form of the decoherence functional

Having identified the distinct notions of time structure in histories theory, we now show how they are manifested in the assignment of probabilities. In the case of discrete time, given the histories Hilbert space \mathcal{V} one can define the boundary Hilbert space $\partial\mathcal{V}$ consisting of bounded maps from the initial time Hilbert space H_{t_i} to the final time Hilbert space H_{t_f} . Then the decoherence functional can be written as [10]

$$d(\alpha, \alpha') = \text{Tr}_{\partial\mathcal{V}} (c(\alpha)c^\dagger(\alpha')) \quad (\text{II. 46})$$

where $c(\alpha)$ is an operator on $\partial\mathcal{V}$ defined by

$$c(\alpha) = \text{Tr}_{\mathcal{V}} (\mathcal{A} S \mathcal{U}^\dagger \alpha \mathcal{U}) \quad (\text{II. 47})$$

\mathcal{S} and \mathcal{U} are operators on \mathcal{V} containing the kinematical/causal and dynamical structure respectively. They are defined in the discrete-time case as

$$\mathcal{S}|v_{t_1}\rangle|v_{t_2}\rangle\cdots|v_{t_n}\rangle = |v_{t_n}\rangle|v_{t_1}\rangle\cdots|v_{t_{n-1}}\rangle \quad (\text{II. 48})$$

$$\mathcal{U} = U(t_1) \otimes U(t_2) \otimes \cdots \otimes U(t_n) \quad (\text{II. 49})$$

The physical meaning of these operators in the decoherence functional also highlights the temporal structure of history theories. The operator \mathcal{S} incorporates the contribution of the system's dynamics to the assignment of probabilities. Its action is to transform a Schrödinger picture operator to a Heisenberg picture one, and it does not change the time label t . The operator \mathcal{S} plays essentially the role of a connection, identifying the structures of Hilbert space at successive instants of time. It is therefore closely related to the Liouville operator. In fact, \mathcal{S} plays a role which is associated with “wave-packet reduction” in standard canonical quantum theory: it essentially forces the single-time projectors that form a history to be multiplied (when forming the class operator \tilde{C}) in the order provided by the partial ordering of time.

The operator \mathcal{A} is a linear map from \mathcal{V} to $\partial\mathcal{V}$. It incorporates the effects of initial conditions (or, if appropriate, final ones also). In the case of discrete time, if $\{|v_{t_1}\rangle\cdots|v_{t_n}\rangle\}$ denotes an orthonormal basis of the history Hilbert space, the matrix elements of \mathcal{A} are given by

$$\langle v'_{t_1}\cdots v'_{t_n}|\mathcal{A}^{ij}|v_{t_1}\cdots v_{t_n}\rangle = \frac{(\rho_f^{1/2})_{v'_{t_1}i}(\rho_0^{1/2})_{jv_{t_1}}}{\text{tr}(\rho_f\rho_i)}\delta_{v'_{t_2}v_{t_2}}\cdots\delta_{v'_{t_n}v_{t_n}} \quad (\text{II. 50})$$

where the indices i, j label the orthonormal basis $\{|v_{t_1}\rangle\otimes|v_{t_n}\rangle\}$ on the boundary Hilbert space.

The above expressions are valid for the discrete-time case as can be checked by direct substitution and with Eq. (I. 1). With care, they can be generalised to the case of continuous time. The operator \mathcal{S} can be constructed from Eq. (II. 48) by going to the continuum limit. The operator \mathcal{U} cannot readily be defined as a continuous-time tensor product of constituent operators; however, Eq. (II. 49) suggests that it should be identified with $e^{-iH_{id}}$, where $id(t) = t$. And, in the case of the operator \mathcal{A} , one can keep a definition similar to (II. 50) with a particular choice of coherent-state basis ² (for an example see reference [10]).

²We have to take into account the fact that, for continuous time, the boundary Hilbert space cannot strictly speaking be embedded into \mathcal{V} , as is possible in the case of discrete time. Nonetheless, it can still be considered as consisting of two copies of the single-time Hilbert spaces. This is the most conservative position, but it is important to note that in the presence of continuous time the initial state might be encoded in more general types of object than density matrices defined at sharp moments of time.

III Parameterised systems: the classical case

III.1 Canonical treatment

The starting point for the Hamiltonian treatment of a constrained system is the phase space Γ of the unconstrained system. A key feature is the existence of a first-class constraint $h(x, p) = \xi$, where ξ is a constant³. By this we mean that the physical system is constrained to lie in the appropriate subset of Γ : namely, the constraint surface C . What is special in a parameterised system is the fact that the Hamiltonian is itself a multiple of $h(x, p) - \xi$. Hence it vanishes on the constraint surface.

The fact that the constraint is first class means that the function h is a generator of a ‘symmetry’ of the system. By this we mean that two points of the constrained surface that are related by a canonical transformation generated by h correspond to the same physical state of the system. In this context we note that the constraint surface is not itself a symplectic manifold since the restriction of the symplectic form to C , ω_C , is degenerate. This means that the points in C are not in one-to-one correspondence with the physical states of the system. The true degrees of freedom lie in the *reduced phase space*. This is constructed as follows.

We first identify the orbits of the one-parameter group of canonical transformations generated by the constraint. The set of all such orbits on C is the reduced phase space Γ_{red} . We denote by $c_p(s)$ the curve obtained when the one-parameter group of canonical transformations generated by h acts on $p \in \Gamma$. Clearly $c_p(0) = p$. Then we define the equivalence relation \sim as follows

$$p \sim p' \text{ for } p, p' \in \Gamma \text{ if there exists } s \in \mathbb{R} \text{ such that } c_p(s) = p'. \quad (\text{III. 1})$$

This relation partitions Γ into orbits. The fact that h is a first-class constraint implies that all orbits with a point in the constraint surface lie wholly in the constraint surface. We can therefore define the reduced phase space as $\Gamma_{red} = C / \sim$. The symplectic form that is naturally defined on Γ_{red} is non-degenerate, since the vector field that generates the action of the constraints lies in the degenerate direction of ω_C .

III.2 The classical problem of time

The above prescription for the construction of the reduced phase space holds for a general constrained system. But when the system is parameterised there is a conceptual difficulty concerning the description of time evolution in Γ_{red} . First, the elements of reduced phase space are themselves solutions to Hamilton’s equation, since the Hamiltonian is proportional to the constraint. On the other hand, a point of the reduced phase space ought to correspond to a state of a system at an instant of time. This is not the case for a parameterised system.

³We make the choice of ξ so that the function h does not contain any constant terms. This will be convenient in the histories treatment.

Indeed, a point of the reduced phase space is a whole *history* and is not restricted to a single moment of time. This means that we do not know what time evolution means in these systems, *i.e.*, what we mean when we say that a point of Γ_{red} ‘evolves’ in time.

Parameterised systems are therefore intrinsically timeless: this is not only because their Hamiltonian vanishes—this would just imply that the dynamics is trivial. But even at the *kinematical* level of the definition of what is meant by time evolution there exists an ambiguity. We cannot think of the basic observables of a parameterised system as quantities that have even the *potentiality* of evolving.

This is the classical ‘problem of time’. One way of tackling this is to impose a gauge-breaking condition that selects a time variable. This essentially amounts to selecting a degree of freedom as the ‘clock’ of the theory. The most elegant implementation of this idea is due to Rovelli [16, 17]. A gauge is chosen in such a way that a function on the constraint surface (that does not commute with the constraint) is identified as a physical clock of the system. One then can assign a time parameter as the value of the associated observable.

There are many problems with this line of thought. First, the choice of the clock variable involves an arbitrary choice of gauge that cannot be justified *a priori*, and when going to the quantum version of this system there is no guarantee that the physical results will be independent of the choice. Secondly, the topology of the constraint surface might not allow us to find a variable that can play the role of a good clock: *i.e.*, one that does not repeat the same values of time in the course of evolution. For example, this is the case when the constraint surface is compact [16] since it cannot have the whole real line \mathbb{R} as a submanifold.

Also, Hartle has severely criticised the notion that clock time is fundamental [18]. His argument is mainly quantum mechanical: that arbitrary choices of time function cannot reproduce the results of standard quantum theory since they might put possible events in a *different* causal relation. Overall, it seems that a ‘clock’ time—all that is available in the canonical description of a parameterised system—might provide a numerical measure of change, but it is unable to incorporate the causal aspect of time in which it is expected to be a partial ordering that determines the succession of events.

From the perspective of the histories programme, the origin of this problem is clear. The canonical approach conflates the two conceptually distinct aspects of time, within the guise of Hamiltonian time evolution. Once they are distinguished the problem is immediately resolved. Histories are paths, and observables are labelled by the external variable t , which contains the notion of partial ordering. When the constraints are implemented, the reduced objects are still histories labelled by the same parameter. In parameterised systems it transpires that the precise value of t is not important: all that matters is that the causal ordering is preserved. This means that the theory is invariant under reparameterisations of time.

We can still define clocks (or construct clocks in practice) as we do in classical mechanics by picking some observable whose values label the changes undergone

by the system. But clock time does not exhaust the properties of time—it is just a quantification of change. Of course, the value that specifies the duration of a given process is not arbitrary but depends on the structure of the physical system (including the clock) we are studying: *i.e.*, its Hamiltonian.

The histories perspective towards time is similar (but not identical) to the Aristotelian distinction between temporality as change and process (*kinesis*, *physis*) and time as measure of this process (*chronos*).

III.3 Histories perspective

III.3.1 The space of paths

We shall now describe the parameterised system in the histories language. As discussed in Section II, the fundamental object is the space of classical histories Π , consisting of paths $\gamma : \mathbb{R} \rightarrow \Gamma$. This can be equipped with a symplectic form Ω which can be written in terms of the canonical coordinates x_t^i and p_t^j as

$$\Omega = \int dt \sum_i dp_t^i \wedge dx_t^i \quad (\text{III. 2})$$

Equivalently, the fundamental Poisson bracket on Π is

$$\{x_t^i, p_{t'}^j\} = \delta^{ij} \delta(t, t') \quad (\text{III. 3})$$

The functions x_t^i are a special case of a construction of families of observables labelled by t . That is, given a function f on Γ one defines a family of functions F_t on Π as

$$F_t(\gamma) := f(\gamma(t)) \quad (\text{III. 4})$$

III.3.2 Functions on Π

Any function H on Π generates a one-parameter group of canonical transformations. We shall denote this as $s \mapsto T_H(s)$, and its action by automorphisms on the algebra of functions F on Π as

$$F \rightarrow T_H(s)[F]. \quad (\text{III. 5})$$

In particular, the functions that generating time translations— V , H_κ and S_κ —are naturally defined on Π as

$$V = \sum_i \int dt p_{it} \dot{x}_{it} \quad (\text{III. 6})$$

$$H_\kappa = \int dt \kappa(t) h(x_t, p_t) \quad (\text{III. 7})$$

$$S = V - H_\kappa \quad (\text{III. 8})$$

Now H_κ is the generator of the symmetry of the system. As $\kappa(t)$ varies, H_κ form the generators of an infinite-dimensional Abelian group that acts on Π by the canonical transformations $T_{H_\kappa}(s)$.

III.3.3 The reduction procedure

We now restrict ourselves to the paths on the constraint surface by imposing $h(x_t, p_t)(\gamma) = \xi$ for all t . In the special case that $\xi = 0$ this is equivalent to the condition that $H_\kappa = 0$ for all κ , since the set of all measurable functions separates \mathbb{R} . By this means, the history constraint surface C_h is defined as the space of maps from \mathbb{R} to C .

The next step is to study the orbits of the group of constraints on Π . We define the equivalence relation \sim in Π as follows:

$$\gamma_1 \sim \gamma_2 \text{ if there exists a measurable function } \kappa \text{ such that } \gamma_1 = T_{H_\kappa}(s)\gamma_2.$$

Then the quotient $C_h / \sim := \Pi_{red}$ is the space of equivalence classes of paths. Its elements can be written as $\tilde{\gamma} = [\gamma]$, *i.e.*, the equivalence class which contains γ .

The transformations $T_{H_\kappa}(s)$ generated by H_κ preserve the t label in a path, since for any family $\{F_t \mid t \in \mathbb{R}\}$ defined by Eq. (II. 29) we have

$$\{H_\kappa, F_t\} = \kappa(t)G_t. \quad (\text{III. 9})$$

The function G_t is defined by (II.3) in terms of the function $g \in C^\infty(\Gamma)$ given by

$$g = \{h, f\}. \quad (\text{III. 10})$$

Therefore Π_{red} is identical to the space of paths on the reduced phase space, *i.e.*, maps $\mathbb{R} \rightarrow \Gamma_{red}$. It also inherits a natural symplectic structure Ω_{red} . It is a standard theorem of symplectic reduction that for any function F on Π that commutes weakly with H_κ ⁴ there corresponds a unique function \tilde{F} on Π_{red} . This is constructed as follows. If F commutes with H_κ it is constant on its orbits; hence if we denote an orbit as $[\gamma]$, we can define

$$\tilde{F}([\gamma]) = F(\gamma) \quad (\text{III. 11})$$

It is a standard result that the reduction preserves the Poisson brackets:

$$\{\widetilde{F}, \widetilde{G}\} = \{\tilde{F}, \tilde{G}\}. \quad (\text{III. 12})$$

In particular, this holds for the families $\{F_t \mid t \in \mathbb{R}\}$ defined earlier.

III.3.4 The action principle

In the absence of constraints the paths that satisfy the equations of motion are obtained by the requirement

$$\{S, F\}(\gamma) = 0 \quad (\text{III. 13})$$

for all F .

⁴This means that the commutator $\{H_\kappa, F\}$ vanishes on the constraint surface.

A similar condition could be imposed in the present case. But it is necessary to reduce the action to a function on Π_{red} . This is possible only if it is compatible with the equivalence relation \sim generated by the group of constraints. This implies that

$$[T_{S_\kappa}(s)(\gamma)] = [T_{S_\kappa}(s)(\gamma')] \quad (\text{III. 14})$$

if $\gamma \sim \gamma'$. Therefore, there should exist a function $\lambda'(\cdot)$ such that

$$T_{S_\kappa}(s)T_{H_\lambda}(s') = T_{H_{\lambda'}}(s')T_{S_\kappa}(s) \quad (\text{III. 15})$$

for all functions $\kappa(\cdot)$, $\lambda(\cdot)$ and $s, s' \in \mathbb{R}$.

We have that $\{S_\kappa, H_\lambda\} = H_{\dot{\lambda}}$, which indeed implies that by defining λ' as $\lambda'(t) = \lambda(t + s)$, equation (III. 15) is satisfied. Hence S_κ can be projected to a function \tilde{S} on Π_{red} which will determine the classical paths as the elements γ of Π_{red} that satisfy

$$\{\tilde{S}, F\}(\gamma) = 0 \quad (\text{III. 16})$$

for all functions F .

In fact, \tilde{S} contains only the Liouville part since it transforms all families \tilde{F}_t defined by (II. 29) to \tilde{F}_{t+s} . This comes from the fact that

$$\{\tilde{S}, \tilde{F}_t\} = \{\widetilde{S}, \widetilde{F}_t\} = \{V - \widetilde{H_\kappa}, F_t\} = \{\widetilde{V}, \widetilde{F}_t\} = \{\tilde{V}, \tilde{F}_t\}. \quad (\text{III. 17})$$

III.3.5 Reparameterisation invariance

The Liouville operator on the reduced phase space determines the classical solutions. These are characterised by reparameterisation invariance, whose generator is a smeared version of the Liouville operator

$$V_\lambda = \int dt \lambda(t) p_t \dot{x}_t. \quad (\text{III. 18})$$

This satisfies

$$\{V_\lambda, H_\kappa\} = H_{\dot{\lambda}\kappa + \dot{\kappa}\lambda} \quad (\text{III. 19})$$

and therefore can be projected to a function \tilde{V}_λ on Π_{red} . Clearly we have

$$\{\tilde{V}, \tilde{V}_\lambda\} = V_{\dot{\lambda}} \quad (\text{III. 20})$$

The action of V_λ is as follows. If we define

$$\tau_\lambda(t) := \int_a^t \frac{ds}{\lambda(s)} \quad (\text{III. 21})$$

(a is an arbitrary number) we can reparameterise the paths from the variable t to the variable τ as long as $\tau_\lambda(t)$ is a strictly increasing function of t . We can then take the inverse $t(\tau)$. This means we write $\gamma'(\tau) = \gamma(t(\tau))$. Then we can write the family of functions

$$F_\tau := f(\gamma'(\tau)) = f(\gamma(t(\tau))) \quad (\text{III. 22})$$

in analogy to Eq. (II. 29). Clearly then

$$V_\lambda = \int d\tau p_\tau \frac{dx_\tau}{d\tau} \quad (\text{III. 23})$$

The action of the one-parameter group of transformations V_λ is then

$$F_\tau \rightarrow F_{\tau+s}. \quad (\text{III. 24})$$

This means that V_λ generates translations in the reparameterised time variable τ . Or equivalently

$$F_t \rightarrow F_{\tau_\lambda^{-1}(\tau_\lambda(t)+s)} \quad (\text{III. 25})$$

This equation implies that for the classical paths (*i.e.*, those invariant under the action of the Liouville function) are left invariant under the action of the smeared Liouville function. Hence reparametrisations of time leave invariant the solutions to the classical equations of motion.

In order for τ_λ to be strictly increasing, its derivative has to be everywhere positive and non-zero; hence $\lambda(t)$ has to be positive and non-zero. Then, for each $\lambda(\cdot)$, V_λ generate a semigroup⁵ of canonical transformations on Π , with a corresponding \tilde{V}_λ on Π_{red} which generate translations in reparameterised time under which the classical solutions are invariant.

III.4 Harmonic oscillators with constant energy difference

We shall now give an example of the construction described above. It is a parameterised system that has been extensively studied, because its quantisation is particularly simple. This consists of two harmonic oscillators constrained to have a constant energy difference.

Here the phase space is \mathbb{R}^4 . It is spanned by the global coordinate system x_0, x_1, p_0, p_1 . The basic Poisson bracket is

$$\{x_i, p_j\} = \delta_{ij} \quad (\text{III. 26})$$

where $i, j = 0, 1$. We also have the first-class constraint $h = \frac{1}{2}(p_0^2 + x_0^2 - p_1^2 - x_1^2) = E$. The Hamiltonian of the system is proportional to $h - E$.

It is more convenient to use complex coordinates $w_0 = (x_0 + ip_0)/\sqrt{2}$ and $w_1 = (x_1 + ip_1)/\sqrt{2}$ so that

$$h = w_0^* w_0 - w_1^* w_1 = E. \quad (\text{III. 27})$$

In the history version, a path $\gamma \in \Pi$ is parameterised by the coordinates $(x_{0t}, p_{0t}, x_{1t}, p_{1t})$. The space Π carries a symplectic form

$$\Omega = \int dt \sum_{i=0}^1 dp_{it} \wedge dx_{it} \quad (\text{III. 28})$$

⁵It lacks inverses because $\lambda(t)$ must be positive

The corresponding Poisson bracket is

$$\{x_{it}, p_{jt'}\} = \delta_{ij} \delta(t, t'). \quad (\text{III. 29})$$

We can also coordinatise Π with (w_{0t}, w_{1t}) . In terms of these, the symplectic form reads

$$\Omega = i \int dt dw_{0t}^* \wedge dw_{0t} + dw_{1t}^* \wedge dw_{1t} \quad (\text{III. 30})$$

The action of H_λ on Π is given by

$$(w_{0t}, w_{1t}) \rightarrow (w_{0t} e^{-i\lambda(t)}, w_{1t} e^{i\lambda(t)}) \quad (\text{III. 31})$$

We use coordinates a_t, ϕ_t, χ_t on the constraint surface that are defined by

$$w_{0t} := \sqrt{E} \cosh a_t e^{i\phi_t} \quad (\text{III. 32})$$

$$w_{1t} := \sqrt{E} \sinh a_t e^{i\chi_t} \quad (\text{III. 33})$$

The degenerate two-form Ω_C then reads

$$\Omega_C = -E \int dt \sinh 2a_t (da_t \wedge d\phi_t + da_t \wedge d\chi_t) \quad (\text{III. 34})$$

The parameters $\psi_t := (\phi_t + \chi_t)/2$ and a_t are constant on the H_κ orbits, hence they can be used as coordinates on the reduced manifold. The coordinates $\zeta_t := (\phi_t - \chi_t)/2$ correspond to the degenerate directions of Ω_C . We can therefore parameterise Π_{red} by ψ_t and $c_t = E \cosh 2a_t$, and write the symplectic form Ω_{red} in the local coordinate form

$$\Omega_{red} = \int dt d\psi_t \wedge dc_t. \quad (\text{III. 35})$$

The action S_κ acts on Π as

$$(w_{0t}, w_{1t}) \rightarrow (w_{0t+s} \exp(-i \int_0^s \kappa(t+s') ds'), w_{1t+s} \exp(i \int_0^s \kappa(t+s') ds')) \quad (\text{III. 36})$$

which on Π_{red} projects as

$$(\psi_t, c_t) \rightarrow (\psi_{t+s}, c_{t+s}) \quad (\text{III. 37})$$

which identifies the elements $\tilde{\gamma}$ of Π_{red} as the ones satisfying

$$\psi_t(\tilde{\gamma}) = \text{const.} \quad (\text{III. 38})$$

$$c_t(\tilde{\gamma}) = \text{const.} \quad (\text{III. 39})$$

Clearly the Liouville function on Π_{red} is equal to

$$\tilde{V} = \int dt c_t \dot{\psi}_t. \quad (\text{III. 40})$$

IV The quantum treatment

IV.1 Reduced phase space quantisation

The logic of reduced phase space quantisation is to focus on the reduced phase space and forget the classical procedure by which one arrived there. Then one can use some quantisation algorithm to construct a quantum theory, that will contain observables corresponding to ones defined on the reduced phase space. In this approach one has to identify the temporal structure already at the classical level (for example, through a time function as explained in section III.2). This means we would inherit the problems associated with lack of gauge invariance and loss of temporal ordering that characterise the classical problem of time.

In the histories quantisation one should first look for a history group on Π_{red} . This will be in general different from the Weyl group, and its structure will depend on the topology of the reduced phase space. Then we should construct the history Hilbert space, by requiring that it carries a representation of the history group.

In general Π_{red} has a non-trivial topology. This implies that the choice of the history group is complicated, and for this reason we have chosen to work with a Dirac quantisation scheme.

IV.2 Dirac quantisation

In the canonical quantisation scheme the Dirac method consists in finding the Hilbert space that carries a unitary representation of the appropriate canonical group. We further require that the classical constraint is represented by a self-adjoint operator h . Then the physical Hilbert space is the linear subspace of the Hilbert space that corresponds to the ξ -eigenvalue of the constraint operator h .

IV.2.1 The quantum mechanical problem of time

Even if we ignore the technical problems of the case where the constraint has a continuous spectrum, the Dirac formalism in the canonical setting suffers from the problem of time. As emphasised earlier, there is no natural notion of time evolution in the physical Hilbert space, since the Hamiltonian vanishes there. As such we cannot speak about time evolution either of states or of observables in the physical Hilbert space. A possibility that has been proposed in the canonical context is that time might be identified with some quantum mechanical observable. This is again a notion of ‘clock time’. This is even more difficult to accept in quantum theory, because any physical clock inevitably undergoes quantum fluctuations, and there is no guarantee that it would be able to respect the temporal ordering of events. To this, one can add the fact that the quantum theory would be explicitly non gauge-invariant.

One proposal that partially remedies this problem, is that clock time is meaningful only in a classical limit; for instance when the wave function describing

the system is of WKB type. (This is very prominent in the discussions of the Wheeler-DeWitt equation in quantum cosmology.) This fact removes the gauge non-invariance argument since time *emerges* only for a particular class of states, and the full quantum theory is fully invariant. But this approach again leads to problems: There is no *a priori* reason why the semiclassical states that have a classical notion of time should be selected as special. It is usually pointed out that classicality is necessary in quantum cosmology (a parameterised system) since we observe a classical spacetime, but of course this does not constitute an explanation. Sometimes the notion of decoherence is evoked, but again this depends on an *a posteriori* split between system and environment that cannot be justified from a first principles knowledge of quantum theory. But the more severe criticism is one that proceeds analogous to according to that in the classical case: If time is to be viewed only as a clock (semiclassical now rather than classical), the notion of temporal succession is likely to be compromised, for it is not encoded in the structure of the clock.

As we will see this problem is solved when the Dirac procedure is implemented in the *HPO histories* scheme.

IV.2.2 The quantisation algorithm

We shall follow the procedure we have described in section II for unconstrained systems. That is, we identify the history group from a study of the space Π of classical histories. Then we seek a representation of this group such that the Hamiltonian constraint H_κ is a well-defined self-adjoint operator. Then the group of constraints generated classically by H_κ for all measurable functions $\kappa(\cdot)$ is represented in the history Hilbert space by the unitary operators e^{-iH_κ} .

In accordance with our classical analysis, the group of constraints is a symmetry of the system. This means that a projection operator P corresponds to physically the same proposition as $e^{iH_\kappa} P e^{-iH_\kappa}$. This correspondence ought to be reflected in the assignment of probabilities, and hence in the decoherence functional. One has to demand that

$$d(e^{iH_\kappa} \alpha e^{-iH_\kappa}, e^{iH_\lambda} \beta e^{-iH_\lambda}) = d(\alpha, \beta) \quad (\text{IV. 1})$$

for all pairs of projection operators α, β and measurable functions $\kappa(\cdot), \lambda(\cdot)$.

This condition is implemented by substituting α with $\mathbf{E} \alpha \mathbf{E}$ in the expression for the decoherence functional, where \mathbf{E} is the projector onto the closed linear subspace where H_κ takes values zero for all functions $\kappa(\cdot)$. Clearly the property IV. 1 is satisfied since $e^{iH_\kappa} \mathbf{E} = \mathbf{E}$. This is the natural condition from the perspective of Dirac quantisation. It implies that only the gauge-invariant part of the projector is relevant to the probability assignment. The range of \mathbf{E} is the Hilbert space of ‘physical’ histories. Nonetheless, it is not necessary to restrict our description on observables living on this space. Any propositions in \mathcal{V} is acceptable in the quantum theory. But only its gauge-invariant part contributes to the probability assignment.

The projector \mathbf{E} can be heuristically considered as the continuous analogue of a tensor product $\otimes_t E_t$, where E_t is the projector onto the physical subspace

at time t . In a discrete time version this amounts to writing the class operator that appears in the decoherence functional as

$$C_\alpha = E\alpha_{t_1}(t_1)E \dots E\alpha_{t_n}(t_n)E = E\alpha_{t_1}E \dots E\alpha_{t_n}E \quad (\text{IV. 2})$$

Note that the presence of the projectors E implies that the Heisenberg-time dependence does not affect the final expression.

Since \mathbf{E} remains unaffected when acted upon by e^{-iH_κ} , in the final expression of the decoherence functional the result is the same as substituting a reduced expression for the \mathcal{S} operator in the decoherence functional (together with dropping \mathcal{U} ; indeed $\mathcal{U} = e^{-iH_{id}}$ just performs a constraint transformation which leaves the projector \mathbf{E} invariant).

For the operator \mathcal{S} we have $\mathcal{S}_{red} = \mathbf{E}\mathcal{S}\mathbf{E}$, or in a discretised form

$$\mathcal{S}_{red}(|v_{t_1}\rangle \dots |v_{t_n}\rangle) = E|v_{t_n}\rangle E|v_{t_1}\rangle \dots E|v_{t_{n-1}}\rangle \quad (\text{IV. 3})$$

Finally, time translations which are represented by the action operator $S_\kappa = V - H_\kappa$, are now to be implemented by

$$S_{red} = \mathbf{E}S_\kappa\mathbf{E} = \mathbf{E}V\mathbf{E}. \quad (\text{IV. 4})$$

An important property of the Liouville operator is that it leaves the decoherence functional invariant. Namely, if we consider the transformation

$$\alpha \rightarrow \alpha' = e^{isV}\alpha e^{-isV} \quad (\text{IV. 5})$$

then

$$d(\alpha', \beta') = d(\alpha, \beta) \quad (\text{IV. 6})$$

A simple proof of this, follows from the remark that the one-parameter group generated by the Liouville operator translates the temporal support of history propositions. Hence for a homogeneous projector, written formally as $\alpha = \otimes_t \alpha_t$, the transformation (IV.5) can be thought of as $\alpha \rightarrow \alpha' = \otimes_t \alpha_{t+s}$. Now the projection operator appears in the decoherence functional through the class operator \tilde{C} . In the case of the parameterised system, the class operator does not depend on the Hamiltonian: rather it depends only on the ordering of the single-time projectors α_t , which is not affected by the transformation (IV.5). From this (IV.6) follows.

This implies that the kinematical time translations are a genuine symmetry of the system. They correspond to an invariance under parameterisations of the real axis \mathbb{R} of time. Actually, the most general symmetries of the quantum system are the quantum version of the transformations generated by V_λ . The operator V_λ generates the transformations

$$\otimes_\tau \alpha_\tau \rightarrow \otimes_\tau \alpha_{\tau+s} \quad (\text{IV. 7})$$

where $\tau(t) = \int_a^t ds/\lambda(s)$. Clearly when $\kappa(t) > 0$, $\tau(t)$ is a strictly increasing function of t , and the ordering is preserved. Hence with the same argument as before we get

$$d(e^{isV_\lambda}\alpha e^{-isV_\lambda}, e^{isV_{\lambda'}}\beta e^{-isV_{\lambda'}}) = d(\alpha, \beta) \quad (\text{IV. 8})$$

This is the sense in which the probability assignment is reparameterisation invariant. In the example given below, this construction will be made explicit.

This are the general guidelines for the quantisation procedure of parameterised systems. They may need to be modified or augmented in individual cases.

An instance of this is the case where $\xi \neq 0$. We cannot then take the ξ eigenspace of H_κ as providing the necessary projector to enforce the Dirac procedure. Rather we should return to a discretised version and consider the projectors E at the ξ -eigenspaces of the single-time constraint operators, and from them construct a suitable expression for the continuous analogue of $\mathbf{E} = E \otimes E \otimes \cdots \otimes E$. The projector \mathbf{E} should again be a spectral projector of the Hamiltonian, if the invariance condition (IV.1) is to hold. We shall give an example of this construction later, so we do not further elaborate here.

We should also remark that a knowledge of the single-time projectors E is important in the construction of the decoherence functional. According to equation (II. 47) one should consider the boundary Hilbert space, where the contribution of the initial and/or final state is contained. This means that we need consider initial and final density matrices lying within the physical Hilbert space as objects entering the definition of the operator \mathcal{A} . Thus we have to impose

$$[E, \rho_0] = [E, \rho_f] = 0 \quad (\text{IV. 9})$$

This is equivalent to reducing the boundary Hilbert space to its physical subspace: the range of the projector $E \otimes E$.

IV.3 The harmonic oscillators with constant energy difference

IV.3.1 Canonical treatment

The canonical commutation relations

$$[x_i, p_j] = \delta_{ij} \quad (\text{IV. 10})$$

can be represented in the Fock space $e^{\mathbb{C}^2}$ on which Hamiltonian $H = \frac{1}{2}(p_0^2 + x_0^2 - p_1^2 - q_1^2)$ is a well-defined self-adjoint operator. The creation and annihilation operators are written in terms of the generators of the canonical group as

$$b_i = \frac{1}{\sqrt{2}}(x_i + ip_i) \quad (\text{IV. 11})$$

The Hilbert space contains unnormalised coherent states $|\exp w\rangle$ where $w = (w_0, w_1) \in \mathbb{C}^2$. The constraint is

$$h = b_0^\dagger b_0 - b_1^\dagger b_1, \quad (\text{IV. 12})$$

and its eigenstates are clearly

$$|n, m\rangle = \frac{(b_0^\dagger)^n}{\sqrt{n!}} \frac{(b_1^\dagger)^m}{\sqrt{m!}} |0\rangle \quad (\text{IV. 13})$$

and satisfy

$$h|n, m\rangle = (n - m)|n, m\rangle. \quad (\text{IV. 14})$$

Clearly, the condition that h takes the value ξ can be realised only if ξ is an integer, say $N = n - m$. The N -eigenstates of h are then $|N + m, m\rangle$ for all positive integers m , and the corresponding projector is

$$E = \sum_{m=0}^{\infty} |N + m, m\rangle \langle N + m, m|. \quad (\text{IV. 15})$$

Its diagonal matrix elements in a coherent state basis are

$$\langle \exp w | E | \exp w \rangle = \sum_{m=0}^{\infty} \frac{1}{\sqrt{n!m!}} (\bar{w}_0 w_0)^{N+m} (\bar{w}_1 w_1)^m \quad (\text{IV. 16})$$

Let us define

$$e^{I_N[a, b]} = \sum_{m=0}^{\infty} \frac{1}{\sqrt{(N+m)!m!}} a^{N+m} b^m \quad (\text{IV. 17})$$

Then

$$I_0[a, b] = ab \quad (\text{IV. 18})$$

and we therefore have

$$\langle \exp w | E | \exp w \rangle = e^{I_N[\bar{w}_0 w_0, \bar{w}_1 w_1]} \quad (\text{IV. 19})$$

IV.3.2 The history space

For the representation of the history group, we seek a Hilbert space \mathcal{F} which can be written as $\mathcal{V} = \otimes_t H_t$ with H_t a copy of the Hilbert space of the canonical theory. The space of smearing functions for the history algebra can be chosen as the set of square-integrable, \mathbb{R}^2 -valued functions on \mathbb{R} , and its complexification is isomorphic to $\mathbb{C}^2 \otimes L^2(\mathbb{R}, dt)$. Following the standard method it is easy to show that

$$\mathcal{V} = e^{\mathbb{C}^2 \otimes L^2(\mathbb{R}, dt)} \quad (\text{IV. 20})$$

The representation of the history group is constructed by writing its generators as combinations of the creation and annihilation operators

$$x_t^i = \frac{1}{\sqrt{2}} (b_t^i + b_t^{i\dagger}) \quad (\text{IV. 21})$$

$$p_t^i = \frac{-i}{\sqrt{2}} (b_t^i - b_t^{i\dagger}) \quad (\text{IV. 22})$$

It follows that the operator H_κ exists as a self-adjoint operator which implements the automorphisms

$$e^{iH_\kappa} b_t^0 e^{-iH_\kappa} = e^{-i\kappa(t)} b_t^0 \quad (\text{IV. 23})$$

$$e^{iH_\kappa} b_t^1 e^{-iH_\kappa} = e^{i\kappa(t)} b_t^1 \quad (\text{IV. 24})$$

The Hilbert space \mathcal{V} contains the natural unnormalised coherent states $|\exp w(\cdot)\rangle$, with $w(\cdot) \in \mathbb{C}^2 \otimes L^2(R, dt)$. This provides the fundamental relation between the Fock space and the continuous tensor product of single-time Hilbert spaces:

$$\begin{aligned} \otimes_t H_t &= \mathcal{V} \\ \otimes_t |\exp w_t\rangle_{H_t} &\rightarrow |\exp w(\cdot)\rangle \end{aligned} \quad (\text{IV. 25})$$

IV.3.3 The spectrum of the Hamiltonian

According to our previous analysis, in order to construct the projector onto the physical subspace we need to study the spectrum of the constraint operator H_κ . The Hamiltonian has the generalised eigenstates

$$|t_1, \dots, t_n; t'_1, \dots, t'_m\rangle := \frac{1}{\sqrt{n!m!}} b_{t_1}^{0\dagger} \dots b_{t_n}^{0\dagger} b_{t'_1}^{1\dagger} \dots b_{t'_m}^{1\dagger} |0\rangle \quad (\text{IV. 26})$$

for which

$$\begin{aligned} H_\kappa |t_1, \dots, t_n; t'_1, \dots, t'_m\rangle = \\ (\kappa(t_1) + \dots + \kappa(t_n) - \kappa(t'_1) - \dots - \kappa(t'_m)) |t_1, \dots, t_n; t'_1, \dots, t'_m\rangle \end{aligned} \quad (\text{IV. 27})$$

In terms of the coherent state vectors they read

$$\langle \exp w(\cdot) | t_1, \dots, t_n; t'_1, \dots, t'_m \rangle = \frac{1}{\sqrt{n!}} \bar{w}^0(t_1) \dots \bar{w}^0(t_n) \bar{w}^1(t'_1) \dots \bar{w}^1(t'_m) \quad (\text{IV. 28})$$

The actual eigenstates are smeared by real functions $\phi^{(n,m)}(t_1, \dots, t_n; t'_1, \dots, t'_m)$, that are separately symmetric with respect to their unprimed and primed arguments. They are elements of $(L^2(\mathbb{R}, dt)_{\mathbb{R}^n})_S \otimes (L^2(\mathbb{R}, dt)_{\mathbb{R}^m})_S$. If we denote

$$|\phi^{(r,s)}\rangle = \int dt_1 \dots dt_n dt'_1 \dots dt'_r \phi^{(r,s)}(t_1, \dots, t_n; t'_1, \dots, t'_r) |t_1, \dots, t_n; t'_1, \dots, t'_r\rangle \quad (\text{IV. 29})$$

then we have

$$\begin{aligned} H_\kappa |\phi^{(n,m)}\rangle &= \int dt_1 \dots \int dt_n dt'_1 \dots dt'_m \\ &\times \left(\sum_{i=1}^n \kappa(t_i) - \sum_{i=1}^m \kappa(t'_i) \right) \phi^{(n,m)}(t_1, \dots, t_n; t'_1, \dots, t'_m) |\phi^{(n,m)}\rangle \end{aligned} \quad (\text{IV. 30})$$

The decomposition of unity in terms of the generalised eigenstates of the constraint operator reads

$$1 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\sqrt{n!m!}} \int dt_1 \dots dt_n dt'_1 \dots dt'_m |t_1, \dots, t_n; t'_1, \dots, t'_m\rangle \langle t_1, \dots, t_n; t'_1, \dots, t'_m| \quad (\text{IV. 31})$$

IV.3.4 The physical subspace

Let us now consider the identification of the projector onto the physical subspace.

The case $\xi = 0$. In this case we are interested in finding the subspace of \mathcal{V} on which H_κ is zero for every κ . From equation IV. 30 it is evident that the (generalised) eigenvalues of the Hamiltonian vanish for all $\kappa(t)$ only when $n = m$ and $t_i = t'_i$ for all $i = 1, \dots, n$. This means that the physical subspace is ‘spanned’ by the generalised eigenvectors of the form $|t_1, \dots, t_n; t_1, \dots, t_n\rangle$. The projector \mathbf{E} is therefore

$$\mathbf{E} = \sum_{n=1}^{\infty} \frac{1}{n!} \int dt_1 \dots dt_n |t_1, \dots, t_n; t_1, \dots, t_n\rangle \langle t_1, \dots, t_n; t_1, \dots, t_n| \quad (\text{IV. 32})$$

and its diagonal matrix elements in the coherent-state basis are

$$\begin{aligned} \langle \exp w(\cdot) | \mathbf{E} | \exp w(\cdot) \rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \\ &\times \int dt_1 \dots dt_n (\bar{w}^0 w^0)(t_1) \dots (\bar{w}^0 w^0)(t_n) (\bar{w}^1 w^1)(t_1) \dots (\bar{w}^1 w^1)(t_n) \\ &= \exp \left(\int dt (\bar{w}_0 w_0 \bar{w}_1 w_1)(t) \right) \quad (\text{IV. 33}) \end{aligned}$$

We remind the reader that due to the analyticity property of the arguments of coherent states, the knowledge of the diagonal matrix elements of an operator is sufficient to determine the operator.

The case $\xi \neq 0$. As we explained earlier the projection operator \mathbf{E} does not correspond to the ξ -eigenvalue of the constraint operator. Rather we should seek to construct \mathbf{E} as a continuous tensor product of single-time projectors E_t into the physical Hilbert space. Each of the single-time projectors E_t is given by equation (IV.15). From equation (IV.25) we see that the matrix elements of \mathbf{E} (a spectral projector of H_κ) can formally be written as

$$\langle \exp w(\cdot) | \mathbf{E} | \exp w(\cdot) \rangle = \prod_t \langle \exp w_t | E_t | \exp w_t \rangle \quad (\text{IV. 34})$$

and then equation (IV.19) suggests strongly that the precise form is

$$\langle \exp w(\cdot) | \mathbf{E} | \exp w(\cdot) \rangle = \exp \left(\int dt I_N [(\bar{w}_0 w_0)(t), (\bar{w}_1 w_1)(t)] \right) \quad (\text{IV. 35})$$

For $N = 0$ the result is in agreement with (IV.33) which was obtained using a different line of reasoning.

IV.3.5 Reparameterisation invariance

Let us now see how the smeared Liouville operator appears in the Hilbert space of the system. One can see that the operator $V_\lambda := \sum_i \int dt \lambda(t) p_{it} \dot{x}_{it}$ generates the automorphisms

$$e^{isV_\lambda} b_w e^{-isV_\lambda} = b_{w'} \quad (\text{IV. 36})$$

where

$$w'(t) := e^{s\kappa(t) \frac{d}{dt}} w(t). \quad (\text{IV. 37})$$

It is easy to see that the definition

$$\tau_\lambda(t) := \int_a^t ds / \lambda(s) \quad (\text{IV. 38})$$

describes the reparameterisation of time, and gives

$$w'(t) = w(\tau_\lambda^{-1}(\tau_\lambda(t) + s)) \quad (\text{IV. 39})$$

in accordance with the classical equation (III.25).

It is easy to check that the projector \mathbf{E} remains invariant under the action of e^{isV_λ} . The decoherence functional is therefore invariant under reparameterisations of time.

V The parameterised relativistic particle

The algorithm we gave for the quantisation of parameterised systems is easily implemented in the case where the constraint operator has a discrete spectrum since the physical ‘subspace’ is a genuine subspace of the Hilbert space. However, this is not the case if the operator has a continuous spectrum, and one then has to modify the algorithm in an appropriate way.

The standard paradigm of a parameterised system with a continuous spectrum for the constraint operator is the relativistic particle. In this section we shall give its classical treatment and then discuss why the method we employed earlier for the quantisation needs to be augmented.

V.1 The classical treatment

The phase space of this system is \mathbb{R}^4 . It is spanned by the global coordinates x_0, x_1, p_0, p_1 , which correspond respectively to the time coordinate, the spatial coordinate, the energy and the momentum of a free relativistic particle in two-dimensional Minkowski spacetime. The phase space is equipped with the Poisson bracket

$$\{x_i, p_j\} = \delta_{ij} \quad (\text{V. 1})$$

where $i, j = 0, 1$. We have the first-class constraint $h = p_0^2 - p_1^2 = m^2$, and the Hamiltonian of the system is proportional to $h - m^2$.

In the history version of this system, a path $\gamma \in \Pi$ is parameterised by the coordinates $(x_{0t}, p_{0t}, x_{1t}, p_{1t})$, and the space Π carries a symplectic form

$$\Omega = \int dt \sum_{i=0}^1 dp_{it} \wedge dx_{it}. \quad (\text{V. } 2)$$

The corresponding Poisson bracket is

$$\{x_{it}, p_{jt'}\} = \delta_{ij} \delta(t, t'), \quad (\text{V. } 3)$$

and the generator of the symmetry is

$$H_\kappa = \int dt \kappa(t) h(x_t, p_t). \quad (\text{V. } 4)$$

The action of H_λ on $\gamma \in \Pi$ is given by

$$(x_{0t}, p_{0t}, x_{1t}, p_{1t}) \rightarrow (x_{0t} + p_{0t} \lambda(t), p_{0t}, x_{1t} - p_{1t} \lambda(t), p_{1t}) \quad (\text{V. } 5)$$

In the constraint surface (defined by $h(x_t, p_t) = m^2$ for all t) one can use the coordinates (x_{0t}, x_{1t}, a_t) where a_t is defined by

$$p_{0t} = m \cosh a_t \quad (\text{V. } 6)$$

$$p_{1t} = m \sinh a_t \quad (\text{V. } 7)$$

and the symplectic form reduces to a degenerate 2-form on C_h

$$\Omega_C = \int m dt (\sinh a_t da_t \wedge dx_{0t} + \cosh a_t da_t \wedge dx_{1t}) \quad (\text{V. } 8)$$

Note that C_h is doubly connected, consisting of a piece with positive and one with negative energy solutions. For simplicity we restrict our description to the former.

The coordinates $\zeta_t = m \sinh a_t x_{0t} + m \cosh a_t x_{1t}$ and a_t are easily seen to be constant on the H_κ orbits, while the coordinates $\xi_t = m \cosh a_t x_{0t} + m \sinh a_t x_{1t}$ correspond to degenerate directions of Ω_C . Then ζ_t and a_t are proper coordinates on Π_{red} , and the symplectic form Ω_{red} is obtained

$$\Omega_{red} = \int dt da_t \wedge d\zeta_t \quad (\text{V. } 9)$$

The action S_κ acts on elements of Π as

$$(x_{0t}, p_{0t}, x_{1t}, p_{1t}) = (x_{0t+s} + p_{0t+s} \int_0^s \kappa(t+s') ds', p_{0t+s}, x_{1t+s} - \int_0^s \kappa(t+s') ds', p_{1t+s}) \quad (\text{V. } 10)$$

which reduces to the action on Π_{red}

$$(a_t, \zeta_t) \rightarrow (a_{t+s}, \zeta_{t+s}), \quad (\text{V. } 11)$$

and can be represented as

$$\tilde{S} = \int dt a_t \dot{\zeta}_t = V_{red}. \quad (\text{V. } 12)$$

The solutions of the classical equations of motion are then all elements $\tilde{\gamma}$ of Π_{red} such that

$$a_t(\tilde{\gamma}) = \text{const.} \quad (\text{V. } 13)$$

$$\zeta_t(\tilde{\gamma}) = \text{const.} \quad (\text{V. } 14)$$

It is interesting to note the form of the solutions to the equations of motion on Π . By eliminating the ‘pure gauge’ variable ξ_t we can write the relation

$$x_{1t} = \frac{-\cosh^2 a_t x_{0t} + \cosh a_t / m}{1 + \sinh a_t \cosh a_t} \quad (\text{V. } 15)$$

For the solutions to equations of motion, a_t and ζ_t are constants. Hence the history variable $-x_{0t}$ can be viewed as a kind of clock time for the system. This means that, for each t , the value of x_{1t} for a classical solution is correlated uniquely to a value of x_{0t} . This value can be taken as measuring the lapse of time from a chosen origin. This choice of time parameter is of course arbitrary and not of physical significance, but it conforms to the standard practice of using the coordinate time as the time parameter of the system. The correspondence between x_{1t} and x_{0t} along the classical solutions is one-to-one at each instant t .

V.2 The need for a non-Fock representation

According to our previous analysis, the history Hilbert space for the relativistic particle should be constructed by considering the representations of the history group $[x_t^i, p_{t'}^j] = i\delta^{ij}\delta(t, t')$ such that the Liouville and the Hamiltonian operators exist

$$V = \sum_i \int dt p_t^i \dot{x}_t^i \quad (\text{V. } 16)$$

$$H_\kappa = \int dt \kappa(t)(p_{0t}^2 - p_{1t}^2). \quad (\text{V. } 17)$$

This implies that the following automorphisms ought to be implemented unitarily

$$p_{it} \rightarrow p_{it} \quad (\text{V. } 18)$$

$$x_{0t} \rightarrow x_{0t} + \kappa(t)p_{0t}s \quad (\text{V. } 19)$$

$$x_{1t} \rightarrow x_{1t} - \kappa(t)p_{1t}s \quad (\text{V. } 20)$$

Unfortunately there does not exist a Fock representation on which these automorphisms can be unitarily implemented.

V.2.1 Discrete time

The standard quantum theory for a parameterised relativistic particle can be readily defined. One can, for example, use the Schrödinger representation for the canonical group on $L^2(\mathbf{R}^2, dx_0 dx_1)$ in which the constraint reads

$$h = -\frac{1}{2} \left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} \right) \quad (\text{V. 21})$$

This has a continuous spectrum. One cannot therefore write a sharp projector onto the value m^2 . It is convenient then to use a regularised projector E_δ where δ is a small number. E_δ is the spectral projector of h in the range $[m^2 - \delta, m^2 + \delta]$. In general as $\delta \rightarrow 0$ the matrix elements of E_δ diverge as δ^{-s} .

Then consider a finite set of n time instants $\{t_i\}$ ($i = 1, \dots, n$) and construct the history Hilbert space $\mathcal{V} = \otimes_{t_i} H_{t_i}$. Clearly the operator

$$e^{-iH\kappa} = \otimes_{t_i} e^{-iht_i \kappa(t_i)} \quad (\text{V. 22})$$

exists (it is the discrete version of the operator \mathcal{U} introduced in II. 49). We can also construct the projection operator

$$\mathbf{E}_\delta = \otimes_{t_i} E_{\delta t_i} \quad (\text{V. 23})$$

As explained earlier in Section IV, the projection onto the physical subspace is equivalent to substituting a reduced Schrödinger operator in the decoherence functional. Hence we can write a regularised reduced Schrödinger operator as

$$\mathcal{S}_\delta(|v_{t_1}\rangle \dots |v_{t_n}\rangle) = E_\delta |v_{t_n}\rangle E_\delta |v_{t_1}\rangle \dots E_\delta |v_{t_{n-1}}\rangle \quad (\text{V. 24})$$

One can therefore write a regularised version of the decoherence functional $d^\delta(\alpha, \beta)$ substituting \mathcal{S}_δ in equation II. 47. As we said earlier, the matrix elements of E_δ diverge as δ^{-r} when $\delta \rightarrow 0$, and hence the renormalised \mathcal{S} should be defined as the weak limit of

$$\mathcal{S}_{ren}(|v_{t_1}\rangle \dots |v_{t_n}\rangle) = \lim_{\delta \rightarrow 0} \delta^{rn} (E_\delta |v_{t_n}\rangle E_\delta |v_{t_1}\rangle \dots E_\delta |v_{t_{n-1}}\rangle) \quad (\text{V. 25})$$

This is equivalent to redefining the decoherence functional for discrete times as

$$d_{ren}(\alpha, \beta) = C \lim_{n \rightarrow 0} \delta^{2rn} d^\delta(\alpha, \beta) \quad (\text{V. 26})$$

This is a finite, generally non-zero number. The coefficient C is introduced here to secure the normalisation condition $d(1, 1) = 1$.

We can therefore define a decoherence functional for all discrete time histories.

The situation for continuous times is very different since it is not in general possible to define a *continuous* tensor product of a family of operators. Instead, one needs to employ a more sophisticated approach to the problem, and this will be the content of a later paper [19].

VI Conclusions

We have shown that the continuous histories programme can accommodate the description of constrained systems using a variant of the Dirac method. In particular, we have seen how the use of an HPO history theory avoids the problem of time that appears classically and quantum mechanically in the canonical treatment. This is a major reason for using the history methods to study systems of this type.

However, there are two distinct issues that remain to be addressed. The first is of a technical nature: extend the quantisation algorithm to deal with systems that have constraints with continuous spectra. This involves finding physically meaningful representations of the appropriate history algebra: these are expected to be not of Fock type [19].

The other issue is of a conceptual type and is related to the interplay between general covariance and the causal structure in general relativity. We have said that histories are viewed as objects that have an intrinsic temporal ordering. On the other hand, in general relativity the causal structure is obtained *after* solving the equations of motion. One might question therefore whether a history description of the theory necessarily violates general covariance.

Indeed this touches in a second, more general, aspect of the problem of time in quantum gravity. How does one know that the spacelike foliation with respect to which the quantum theory is defined remains spacelike at the next time step if the evolution law is not deterministic? We believe we will have a meaningful answer to this question when we examine the implementation of spacetime diffeomorphisms in the histories version of general relativity. This is something we intend to address soon. Here, let us note only that our formalism involves a mixture of both Lagrangian and Hamiltonian techniques. As such it allows both the group of spacetime diffeomorphisms, and the group of constraints, to be represented on the history space by symplectic transformations. For this reason we expect to be able to clarify the relation of these symmetries.

We conclude with two remarks. First, in the simple systems studied in this paper, general covariance is identical to reparameterisation invariance. Indeed, the smeared Liouville operator is the generator of the most general order-preserving diffeomorphisms of the real line. Diffeomorphisms are then structurally distinct from the Hamiltonian constraints and we would expect them to be implemented as transformations on the reduced phase space of general relativity.

Second, the problem of time in the quantisation of general relativity is conceptually distinct from the problem of time in the quantisation of parameterised systems. The latter is an artifact of the canonical approach and, as we have seen, can be solved by using the histories quantisation method. Only a rather conservative modification of the standard quantum mechanical formalism was required to do this. However, the problem of time in quantum gravity is grounded in a deep disparity between the notion of time in quantum theory and general relativity. Quantum theory seems to require an *a priori* notion of causality, while in general relativity causality emerges from the implementation of the dynamics.

The resolution of this issue is likely to require a much more radical change of the quantum mechanical formalism than has been proposed so far.

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